LECTURE 23

One important application of the derivative is to find the optimal solutions to problems. An optimal solution maximises or minimises a certain performance metric, such as total cost of a complicated business strategy, utility function of an economic model, or the amount of drugs delivered in medicine under spurious environment. Many physical questions can be turned into a mathematical one.

First, we must define the **absolute** maximum and minimum, particularly, of the values of a function.

Definition. Let f be a function with domain D. Then f has an **absolute (global) maximum** value on D at a point c if

$$f(x) \leq f(c), \quad \text{for all } x \in D$$

and an **absolute (global) minimum** value on D at a point c if

$$f(x) \geq f(c), \quad \text{for all } x \in D.$$

Maximum and minimum values of f are called **extreme values** of f. Absolute maximum and minimum don't have to exist on every D.

Example. Consider the function $f(x) = x^2$.

D	**Global extrema**
$(-\infty, \infty)$	No global maxima; global minima at $(0,0)$
$[0,2]$	Global maxima at $(2,4)$; global minima at $(0,0)$
$(0,2]$	Global maxima at $(2,4)$; no global minima.
$(0,2)$	No global extrema

Note that the existence of global extrema depends on the domain. When the interval is open on one or both ends, the existence of global extrema becomes a bit more subtle. For example, in the third case, $(0,2]$, since $x = 0$ is not included, the x-values are only getting so close to 0 without touching it. This means, if you claim that there is a global minimum achieved by some c near 0, by a property of the real number system, you can also find another number $d < c$ such that $d^2 < c^2$, which defeats c's global minimum. By this argument, you won't find a global minimum!

However, you note that on the closed interval $[0,2]$, we have both extrema. This in fact is a true statement if we further suppose that the function is continuous and the interval is bounded, in addition to being closed.

Theorem. *If f is continuous on $[a,b]$, then f attains both a global maximum value M and a global minimum value m in $[a,b]$. That is, there are numbers x_1 and x_2 in $[a,b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and*

$$m \leq f(x) \leq M$$

for every other $x \in [a,b]$.

Remark. The requirements that the domain be closed and bounded and the function be continuous are essential for the conclusions to hold. If you lose any one of them, then the conclusion needs not hold. See the 3rd and 4th case in the previous example, where the function is continuous but the interval is not closed (though bounded).

If the interval is closed and bounded, but the function is not continuous, the conclusion doesn't hold. Consider

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

The function is continuous at every point on $[0,1]$ but never achieves a global maximum.

However, not all functions are monotone. There are peaks and valleys, and more importantly, higher peaks and lower valleys. How do we characterise those?

Definition. A function f has a **relative (local) maximum** value at a point c within its domain D if

$$f(x) \leq f(c)$$

for all $x \in D$ in some open interval containing c. A function f has a **relative (local) minimum** value at a point c within its domain D if

$$f(x) \geq f(c)$$

for all $x \in D$ in some open interval containing c.

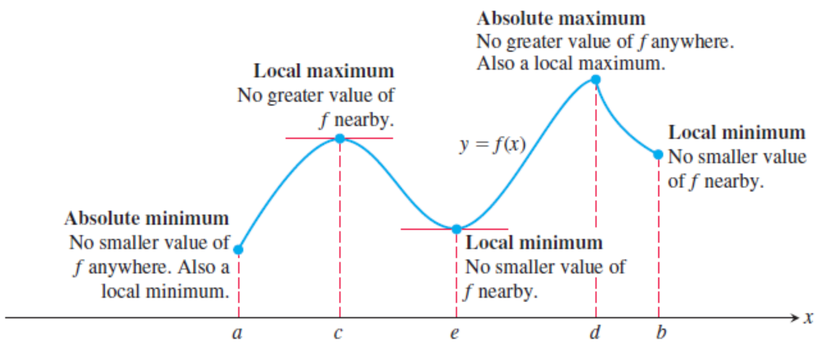

FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

Absolute extremas are local extremas automatically, since in its immediate neighbourhoood, there are no others points that are larger or smaller than it. Therefore, local extremas are in fact a more general notion (a larger set of points), and absolute extremas become special cases.